Learning to ‘deal’: A microgenetic case study of a struggling student’s representational strategies for partitive division

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This paper focuses on the arithmetical understandings and behaviours of one fifteen-year old student with very low attainment in mathematics, as she worked on a sequence of scenario-based partitive division (sharing) tasks with individually-tailored verbal and visuospatial support. The student’s independent and co-created visuospatial representations of arithmetical structures, along with her verbal comments, were analysed qualitatively using a multimodal microgenetic approach. This paper focuses on three particular excerpts which illustrate the fundamentally componential nature of the concept and practice of division, some difficulties that may be experienced when modelling ‘sharing’ tasks, and the pedagogical importance of spatial structuring when a learner is moving between different kinds of representation.

Keywords: Visuospatial representation, multiplicative thinking, numeracy, low attainment, special education

Introduction

I encountered Paula during a larger project investigating low-attaining students’ representational strategies for multiplicative structures. Attending a comprehensive school in inner London, she turned fifteen during the study; however, in certain respects her quantitative reasoning more resembled that of a pre-school child. Her particular stage of arithmetical thinking (struggling with the move from additive to multiplicative reasoning) has been of particular interest to researchers, and her reliance on unitary counting-based strategies is a well-known phenomenon. Atypical for Paula’s age, but common in younger learners, was her heavy use of enactive representation with physical media such as cubes. These several factors, along with the slow progress, provided an excellent opportunity for microanalytic case study: to examine this individual’s arithmetical-representational strategies in fine detail, and note even very small changes taking place. Thus, I focused on an arithmetical concept which the participant did not yet comprehend (division), building on an activity in which she was comfortable (counting), within scenario tasks that allowed for multiple representational variations. This paper presents and discusses some brief but illuminative excerpts from my work with her.

Theoretical background

There is a strong tradition of research into various aspects of early numeracy, such as counting-based arithmetical strategies, taking place in naturalistic teaching/learning environments. Those which focus on children’s own representations of number are often quasi-ethnographic in nature, where (usually very young) children are observed in their mark-making (e.g. Atkinson, 1992) or block-play (e.g. Gura, 1993), and their representations analysed for ‘emergent’ mathematics. Key to this body of work is that it focuses on children’s own, often non-standard, representational strategies; this is in the pedagogical tradition of “de-centring” (Donaldson, 1978), i.e. to shift from an adult perspective and imagine what a scenario, phrase or object might mean to a child. While the details and exact terminology can vary, in psychological research paradigms some kind of representational progression is also generally assumed, moving from the most intuitive/enactive/concrete models of arithmetical relationships, through iconic/pictorial/drawn forms, to the incorporation of abstract symbols and eventual full formal symbolic notation (e.g. Bruner, 1973). The development of mathematical concepts has also been linked to increasing awareness of pattern, and the ability to make connections between one mathematical representation and another, i.e. to notice similarities and differences, is important for a learner’s developing relational thinking. Low attaining students often lack visualisation skills and flexibility, and may indeed find it difficult to replicate and organise representations of groups and patterns (Mulligan, 2011).

Nunes & Bryant (1996), among many others stretching all the way back to Piaget, suggest that to understand multiplication/division represents a significant qualitative change in children's thinking (compared to addition/subtraction) – and so is deserving of particular attention. Regarding the increased complexity, Anghileri (1997) points out that a counting strategy in a multiplication or division task requires three distinct counts: the number in each set, the number of sets, and the total number of items. The second of these – tallying sets rather than units – may be particularly unintuitive for some. Notwithstanding, Carpenter et al.'s (1993) study of kindergarten students (i.e. age 5-6, with <1 year of formal schooling) demonstrated that they could carry out a wider range of division tasks, with greater success, than had formerly been realised – provided the tasks were presented in the form of scenarios which could be directly modelled. Furthermore, they argued that many older students abandon their fundamentally sound problem-solving approaches for the mechanical application of formal arithmetic procedures, and would make fewer errors if they applied some of the intuitive modelling skills of their younger counterparts.

Given this, it is appropriate to combine a subject focus of early division with an analytical focus on informal, nonstandard, and intuitive representational strategies. A previous example is Saundry and Nicol’s (2006) investigation of the drawings young children used in division-based tasks, including a ‘sharing biscuits’ scenario, as used in this study; they describe students manipulating pictures on the page, moving, eliminating, sharing and distributing them, in some cases with patterns of movement resembling the use of physical manipulatives. This is in contrast to much prior research which has analysed visuospatial representations more simply, by organising them into broad categories. However, a third way is possible: considering students’ changing representations via an analytical framework of multiple interrelating aspects (Finesilver, 2014).

Research questions

1. What arithmetical-representational strategies does the student use in division tasks?
2. What do the strategies tell us about their particular weaknesses and capabilities?
3. How do the student’s arithmetical-representational strategies change over time and input?

Methodology

The dataset for this study is taken from a series of four 1:1 problem-solving interviews, each lasting 45 minutes, carried out by the author. While some sessions did include other types of multiplication- and division-based activity (reported elsewhere), a significant proportion of this particular student’s time was given over to the ‘sharing’ tasks described here. It employs microgenetic methods, which were developed for the study of the transition processes of cognitive development (Siegler & Crowley, 1991). They have been widely used in studies of children's arithmetical strategies and particularly in case studies of individuals with difficulties in mathematics (e.g. Fletcher et al., 1998).

Paula had been described in a past Educational Psychologist's report as having "particularly severe" difficulties with numeracy. This was confirmed through classroom observation by the author, discussion with her mathematics teacher, and a 1:1 qualitative assessment of arithmetical and representational capabilities (see Finesilver, 2014). She demonstrated confident ascending counting and writing of two-digit numbers, and appeared to understand the principles of addition and subtraction well (although made frequent errors in practice). This information was used formatively in devising appropriate in situ tasks and support for developing a basic understanding of division.

In each session, Paula was set a series of partitive divisions expressed via the scenario of a given number of biscuits to be shared between a given number of people. The quantities used were two-digit numbers under 30 that divided exactly by 3, 4 or 5. Numbers were chosen in situ, depending on her arithmetical-representational functioning that day and in previous sessions. Representational media were a particular concern to Paula's teacher, as she was approaching high-stakes national examinations where concrete manipulatives would be unavailable. Thus, scenarios and representation types were chosen that allowed working on first with enactive concrete models, then translation of these visuospatial configurations to graphic form. I gave verbal and visuospatial prompts whenever independent activity came to a halt, up to and including co-creating representations with her.

All sessions were audio recorded, all markings on paper collated, and (when it would not interfere with her work) photographs of concrete representations taken. All markings in purple ink are by the researcher. Each task attempt was considered in terms of the thirteen-aspect analytical framework developed in Finesilver (2014), which covers the type of representation created (*media, mode, resemblance*), the relationships between representation and calculation (*motion, unitariness, spatial structuring, consistency, completeness, enumeration, errors, success*), and teacher-student interactions (*verbal* and *visuospatial prompts*). Particular attention was paid to any attempts where change in one or more of the aspects was observed; these are considered microgenetic 'snapshots'.

Data (selected excerpts)

Due to restrictions of space, only a small sample of data may be reproduced in these proceedings; more are included in the accompanying presentation and other publications by the author.

Excerpt 1: Difficulties co-ordinating division requirements, Session 1: 15 ÷ 3 (Figure 1)

Paula having made no independent attempt at the task "fifteen biscuits shared between three people", I give her a pile of 15 cubes. She first pushes them into two roughly but not exactly equal groups. I restate that **three equal** groups are required, and she distributes them into groups of three. I then draw three *unit containers*, and she pushes three cubes into each. I restate the requirement to share out **all** the cubes, and she adds 1-3 more (unequally) to each circle. I ask if the groups are **equal**, and she counts each group, then adjusts, re-counting, until she can present me with three groups of five. She appears to understand my individual comments, but have difficulty co-ordinating the requirements.

Later in this session, I demonstrate the ‘dealing’ process, emphasising the regular repeating motion. Paula is able to replicate this procedure without error, but does not seem confident that the groups will be equal, and counts each to check.

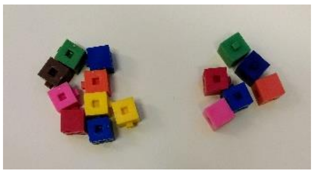	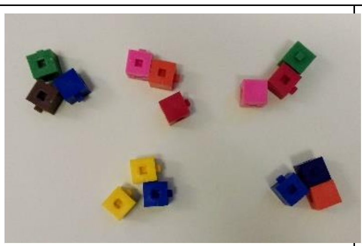	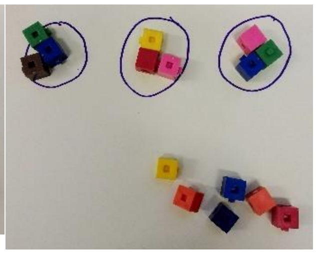	(no photo)
First attempt: “15” “shared” ~~equally~~ ~~between 3~~	*Second attempt: “15” “shared” “equally” ~~between~~ “3”*	*Third attempt: ~~15~~ “shared” “equally” “between 3”*	*Fourth attempt: “15” “shared” ~~equally~~ “between 3”*

Figure 1: Sequence of task attempts; change in *spatial structuring* to introduce unit containers

Excerpt 2: Moving from modelling to drawing, Session 2: 15 ÷ 3, 15 ÷ 5 (Figure 2)

Paula had been given three drawn circles (as previously) and asked to try to complete the sharing without using cubes. Here she first makes 13 dots in each circle (likely intending 15). I draw new circles, and draw 15 dots above (as a non-concrete analogue for the initial pile of cubes in previous tasks). I demonstrate ‘taking’ dots from the ‘pile’ (by crossing them out) and ‘moving’ them (by redrawing them) into the circles. After watching four dots being dealt out in this way, Paula takes over and continues the pattern of motion, successfully adding dots to the circles in a cyclic sequence (apart from one error in the form of an extra dot, subsequently crossed out). She completes the following task independently and without errors.

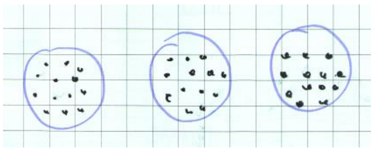	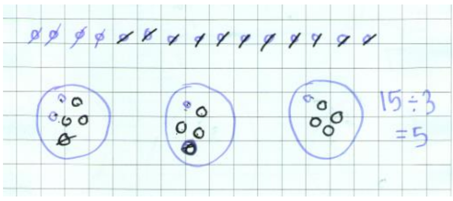	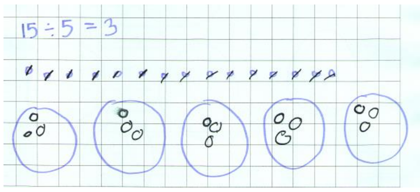
First attempt: 15 (miscounted) ~~shared~~ ~~between~~ “3”	*Second and third attempts: “15” “shared between” “3” with an extra dot (corrected)*	*First attempt: 15 shared into 5 groups of 3*

Figure 2: New mode/media, retaining previous *spatial structuring* and *motion*

Excerpt 3: Confidence in the ‘dealing’ procedure, Session 4: 24 ÷ 4

Paula counts out 24 cubes for herself, and deals them cyclically into four circles. However, she pauses in confusion when almost finished. She places her last cube, counts that group and the one next to it, finds them unequal, and twice moves a cube then re-counts (each time finding three groups of six and one group of five) with dissatisfaction. She looks around and finds the last cube (hidden in her sleeve), allowing her to complete the final group. She verbally states “six each” as her solution.

These three 'snapshot' excerpts are selected to illustrate particular points for discussion regarding components of division and arithmetical-representational strategies.

Discussion

What representational and arithmetical strategies does the student use?

Paula did not initially have any working representation of her own for use in sharing tasks, and exhibited a 'helpless' non-response. While she could read and write number symbols, she could not use them for multiplicative reasoning. However, with a relatively small amount of teacherly input and encouragement, she proved capable of successfully using visuospatial representational strategies to represent equal-groups structures and solve tasks that had previously seemed impossible to her.

Paula's initial preference was for simple modelling with cubes. However, mixed-media/mixed-mode representations (concrete units in drawn containers) were actually most successful, due to the enhanced spatial structuring of the groups provided by the container forms. At first, she distributed and pushed cubes between groups unsystematically, counting to check for equality, adjusting, and recounting. Later, she adopted the more structured 'dealing' procedure.

Paula was willing to move from physical modelling to using fully-drawn representations, with some success. Key to this was keeping both the spatial structuring (i.e. the initial 'pile' and container circles) and the dealing motion (repeating hand movement back and forth between the pile and each of the containers in turn) the same as it had been when modelling with cubes, and emphasising this similarity.

What do the strategies tell us about their particular weaknesses and capabilities?

Initially it could be stated with certainty only that Paula knew the division operation required starting with an initial quantity and separating it into a number of smaller quantities (as this was a consistent response in all attempts). This may seem trivial; however, it is not only a necessary component of division, but may be seen as the most fundamental meaning of 'divide', prior to any notions of dividends, divisors, quotients, or equality.

Paula made two types of error of particular interest in our sessions on partitive division. On eight occasions she broke the rule that groups must be equal (e.g. Excerpt 1, first attempt), and on seven that the initial number of units must be preserved, i.e. no cubes left over, and no increase through taking extra cubes or drawing extra dots (e.g. Excerpt 1, third attempt). Generally either one or the other of these errors occurred, and sometimes correcting one caused the other to occur. This indicates she experienced a tension in trying to satisfy these apparently-competing demands at the same time.

Paula's producing of unequal groups implies either that she did not see it as important for groups to contain an equal number, and/or that she did not know a reliable method for distributing them fairly. The latter is indicated, as when reminded, she counted each individual group and took action to even them up. When group sizes were unequal, they only varied by one or two cubes: it is possible that she considered these groups <u>sufficiently</u> equal. For students who struggle significantly with number, it may seem quite reasonable to treat, say, 20 cubes as a continuous rather than a discrete quantity, and thus to perform an approximate rather than an exact division.

Paula's non-preservation of total units implies either that she did not initially see it as important that all of the initial quantity should be distributed, that she believed that including them in the groups already created would conflict with another requirement of the task, and/or that she had simply forgotten about them. The second interpretation seems most likely, as she distributed the remainder when asked, and then re-counted the group sizes to check for equality. There is a small but highly significant difference in Excerpt 3 (compared to previous task attempts): she realises independently that it is impossible to adjust the groups to make them equal, and deduces there is something wrong – I had specified equal groups, and this is impossible unless there is a missing cube.

These observations together suggest that Paula had a three-part conception of division, corresponding to three independent requirements: separation of the initial quantity into groups, that the groups are of equal size, and that all of the initial quantity have been distributed. The priority relationship between the second two requirements was not constant. For most students using a unit-based concrete model, these three stages would be subsumed into one through 'dealing' units cyclically into groups until all of the initial quantity is gone. It is notable that Paula did not initially do this, instead using an unsystematic distribution process. It seems inconceivable that a 15-year-old in mainstream education has never encountered dealing; however, Paula initially did not independently think to use it in these situations. Furthermore, when first trying dealing, she seemed unconvinced of its reliability in delivering 'fair shares'; this implies not initially connecting the structure of the physical dealing action with the numerical structure, visual pattern, or arithmetical operation. The observation that she later stopped checking by counting (when the deal worked out as expected) implies increasing acceptance of it incorporating the structure of, and thus ensuring, equal groups.

Paula's extreme focus on individual countable units, taken with the instances of her sharing into an incorrect number of groups, indicate the possibility (in line with Anghileri, 1997) that she may have difficulty with the very idea of *groups being countable objects*, i.e. with shifting her focus from unit-level to group-level. This interpretation is consistent with both the fact that my drawn containers were helpful to her (through visually reinforcing groups-as-units), and the fact that, despite this, she was somewhat disinclined to draw them independently.

How do the student's arithmetical-representational strategies change over time and input?

Given the level of support required for working with Paula, it is more helpful to consider the overall content of my input and its effect on Paula, rather than individual instances. In summary, I emphasised the three requirements for 'fair sharing', and introduced a practical method for accomplishing this: dealing. I explicitly encouraged visuospatial unitary representation, and introduced an alternative mode (drawing). These were both influential on Paula's ongoing task strategy choices and behaviours.

Although there was a high number of 'teacher-student' interactions, I followed the principle of keeping each teacherly input minimal. Verbal prompts each related to a specific rule that was broken (e.g. unequal sharing) or a single aspect of the task that was misunderstood (e.g. number of groups). In each case, Paula immediately corrected her error (although sometimes making another while doing so). My visuospatial interactions consisted of drawing containers and demonstration or miming of dealing; in each case, Paula was able to take over, complete the representation and use it to obtain an answer to the division task, and then eventually use the same strategy independently. These kinds of mimicking behaviours may seem trivial to the casual observer, but I argue that for this kind of student it is a significant achievement and important development just to carry out replicatory-structured pattern creation successfully.

While within individual sessions Paula switched from modelling with cubes to drawing, in each subsequent session this temporary confidence had been lost somewhat, and it was necessary to return to the cubes. However, progressively less time was spent in concrete mode, and she also began to draw her own container forms in which to distribute units (examples not included in this paper). It is reasonable to speculate that with further experience, the connections might be strengthened, and the drawn forms regained more quickly and retained for longer.

After I had explicitly demonstrated the dealing process, Paula began increasingly to use this method. While it is true that she required reminding of it in each of the subsequent sessions, she could be observed carrying out the action with increasingly sure and efficient movements. It may be inferred that the repeated success of dealing strengthened her belief in its reliability as a means of fair sharing.

Early on, where Paula had distributed cubes or dots in a disordered way, she often looked at her representation and made adjustments to it via visual approximation, which nevertheless often still resulted in unequal groups. She also presented many incorrect solution representations to me without any attempt to check her work, and simply waited for my response. However, in later sessions, she made attempts to co-ordinate the division requirements herself, for example, checking that all cubes/dots had been distributed and that the resulting groups were equal, addressing these issues if not (e.g. Excerpt 3). Additionally, rather than simply performing a sharing procedure and presenting the representation, she also started to state the group size as her ‘answer’.

Concluding comments

A microgenetic level of analysis of this student’s arithmetical struggles illuminates certain specific difficulties in conceptualising and carrying out division-based tasks which may be unexpected and go unrecognised in classrooms. It also demonstrates the possibility of improvement even in such severe cases, and has pedagogical implications.

Regarding the concept of division, a three-part deconstruction may be seen: (a) the separation of a quantity into a given number of parts, where (b) those parts are equal, and (c) the original quantity is preserved. Also highlighted is the interplay, and potential for tension, between those requirements, or the overriding of one or the other by partial fragmentary understandings. Where there is difficulty considering more than one ‘rule’ at a time, what would have been a simple one-stage calculation thus becomes a complex multi-stage process.

Regarding representational modes, the ease with which Paula switched from concrete to graphic representations of numeric relationships is also significant. Translation between representational modes is commonly considered difficult to achieve, particularly for low-attaining students. It was managed here thanks to carefully-designed scenario tasks and representations that maximised and emphasised correspondences and similarities in visuospatial form and structure, and in hand motion.

The nature and degree of Paula's individual difficulties made progress not only extremely slow and effortful, but uneven and unstable; nevertheless, these excerpts indicate changes, however small when measured against the progress of typically-attaining teenagers. These changes may be considered microprogressions in arithmetical and multiplicative thinking. While it is true that children may sometimes carry out action sequences without understanding their significance, Paula's insecure start but increasingly confident use of dealing – combined with changes in representational mode and accompanying enumeration – indicate a strengthening understanding of the links between the repeated distribution action and the partitioning of quantities into exactly-equal groups. While meaningful symbolic thinking about multiplicative structures may have still been a long way off for Paula, her efforts and achievements in 'learning to deal' deserve appreciation.

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